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STABILITY AND DIMENSION—A COUNTEREXAMPLE TO A CONJECTURE OF CHOGOSHVILI

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ABSTRACT. For every $n \geq 2$ we construct an n-dimensional compact subset X of some Euclidean space E so that none of the canonical projections of E on its two-dimensional coordinate subspaces has a stable value when restricted to X. This refutes a longstanding claim due to Chogoshvili. To obtain this we study the lattice of upper semicontinuous decompositions of X and in particular its sublattice that consists of monotone decompositions when X is hereditarily indecomposable.

1. INTRODUCTION

Let X and Y be subsets of some Euclidean space E. X is said to be removable from Y if for every $\varepsilon > 0$ there is a map $f: X \to E$ with $\|x - f(x)\| < \varepsilon$ so that $f(X) \cap Y = \varnothing$.

In [Ch] Chogoshvili claims that if $X \subset E_m$ (= m-dimensional Euclidean space) is n-dimensional then there exists an $(m-n)$-dimensional affine subspace A of E_m so that X is not removable from A. Moreover, given a coordinate system in E_m, A can be chosen to be parallel to one of the $(m-n)$-dimensional linear subspaces determined by this coordinate system (Remark III, p. 292 in [Ch]).

Recently R. Pol discovered a gap in Chogoshvili's proof which went unnoticed for many years (see [E_1]). Sitnikov's example [Si] demonstrates that the theorem fails if X is not assumed to be compact. The main goal of this article is to present a compact counterexample to the stronger version of Chogoshvili's theorem. We shall prove:

Theorem 1.1. *For every $n \geq 2$ there exists an n-dimensional compact subset X of some m-dimensional Euclidean space E (m depends on n), and a representation $E = \sum_a \bigoplus E_a$ of E as a direct sum of (mutually orthogonal) linear subspaces E_a of the same dimension l each, so that the following holds: for every choice of a basis $\{e_{a,i}\}_{i=1}^{l}$ for E_a, X is removable from every $(m-2)$-dimensional affine subspace of E which is parallel to one of the $(m-2)$-dimensional linear subspaces determined by the basis $\bigcup_a \{e_{a,i}\}_{i=1}^{l}$ of E.*

Remark. By a basis here we mean of course a linear basis and L is an $(m-2)$-dimensional linear subspace determined by the basis if it is the linear span of $m-2$ (i.e. all but 2) of its elements.

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This stronger than just a counterexample to the Chogoshvili claim: first, the affine subspaces from which X is removable are not merely $(m-n)$-dimensional but $(m-2)$-dimensional (even for $n > 2$) and also, as $l > 1$, there is a degree of freedom in the choice of the bases $\{e_{a,i}\}_{i=2}^{l}$ for E_a. Note however that the decomposition $E = \sum_a \bigoplus E_a$ is fixed, so that we do not have a counterexample for the weaker version of the Chogoshvili theorem which remains unsettled. The example, to be constructed in §5, will actually possess even stronger properties which for the sake of convenience will be stated there. In §2 we introduce the concept of n-stable mappings and study their relevant properties. In §3 we present the lattice $\mathrm{DEC}(X)$ of upper semicontinuous decompositions of a compact metrizable space X, and in §4 we study its sublattice $M(X)$ which consists of the monotone decompositions when X is a hereditarily indecomposable continuum.

We give detailed proofs only to results which are applied in the construction. We state without proofs some other results which arise naturally. These are marked by a * and will be proved elsewhere.

The space X in the example is an n-dimensional hereditarily indecomposable (H.I.) continuum. Such spaces were first constructed by Bing [B], and were recently applied by R. Pol [P] to construct subsets of Cartesian product spaces with some remarkable properties.

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2. n-STABLE MAPPINGS

Let $f: X \to Y$ be a mapping. (All spaces in this article are assumed to be metrizable, and all maps are assumed to be continuous.) Recall that a point $y \in Y$ is called a stable value of f if there exists $\varepsilon > 0$ such that for all $g: X \to Y$ with $d(f(x), g(x)) < \varepsilon$ for all x in X, $y \in g(X)$.

Let B_n be a closed n-ball in n-dimensional Euclidean space E_n with boundary S_{n-1}. A mapping $f: X \to B_n$ is *essential* if the map $f/f^{-1}(S_{n-1})$: $f^{-1}(S_{n-1}) \to S_{n-1}$ is not extendable to a map $g: X \to S_{n-1}$. It is well known that a map $f: X \to E_n$ has a stable value if and only if it maps essentially onto some n-ball $B_n \subset E_n$ (i.e. $f/f^{-1}(B_n): f^{-1}(B_n) \to B_n$ is essential).

It is easy to see that the Chogoshvili claim is equivalent to the following: "Let $X \subset E_m$ be n-dimensional. Then there exists an n-dimensional linear subspace L_n of E_m such that the restriction to X of the orthogonal projection $P: E_m \to L_n$ has a stable value." And the stronger version is as above with L_n a subspace determined by the coordinate system and P a coordinate projection.

The following is a more general concept of stability which turns out to be useful.

Definition. A map $f: X \to Y$ is said to be *n-stable* if for every metric space W which contains Y there exists a neighbourhood U of f in $C(X, W)$ so that $\dim g(X) \geq n$ for all g in U. If f is not n-stable then we call it n-unstable.

We apply this definition for compact spaces X, in which case the topology on $C(X, W)$ is induced by the metric $\zeta(f, g) = \sup\{d(f(x), g(x)): x \in X\}$ where d is a metric on W. Through the rest of this section we assume that X is compact.

Proposition 2.1. *A map $f: X \to E_n$ is n-stable if and only if it has a stable value.*

Proof. 1. Let $f: X \to E_n$ have a stable value. Without loss of generality we may assume that f maps essentially onto some n-ball B_n. Let W be a metric space that contains $F(X) = B_n$. Again we may assume that W is a Banach space. There exists a retraction $r: W \to B_n$ so that $\|x - r(x)\| \leq 2d(x, B_n)$, $x \in W$. (See [B.P, p. 61].) Let ε be positive. If there exists some $g_\varepsilon: X \to W$ with $\|f - g_\varepsilon\| < \varepsilon$ and $\dim g_\varepsilon(X) \leq n-1$ then $r: g_\varepsilon(X) \to B_n$ can be approximated by some $r_\varepsilon: g_\varepsilon(X) \to B_n$ with $\|r - r_\varepsilon\| < \varepsilon$ and $\dim r_\varepsilon g_\varepsilon(X) \leq n-1$. It follows that $f_\varepsilon = r_\varepsilon g_\varepsilon: X \to B_n$ satisfies $\|f - f_\varepsilon\| < 5\varepsilon$ and $\dim f_\varepsilon(X) \leq n-1$. For sufficiently small ε however this is impossible since as f maps essentially on B_n, $f_\varepsilon(X)$ must contain some n-ball, and it follows that f is n-stable. □

2. Let $f: X \to E_n$ be n-stable, and let us assume that $f(X)$ is contained in some n-ball B_n. If f has no stable values then by a standard argument for every finite subset F of E_n and every $\varepsilon > 0$ there exists a map $g = g_{F,\varepsilon}: X \to B_n$ with $\|f - g\| < \varepsilon$ and $g(X) \subset B_n \setminus F$. As for every $\varepsilon > 0$ there exist a finite subset F_ε of B_n and a retraction $r_\varepsilon: B_n \setminus F_\varepsilon \to Y$, with $\dim Y \leq n-1$ and $\|x - r_\varepsilon x\| < \varepsilon$, $f_\varepsilon = r_\varepsilon g$ approaches f as close as we please, and $\dim f_\varepsilon(X) \leq n-1$. Hence f must be n-unstable. □

Proposition 2.2*. *A map $f: X \to Y$ is n-stable if and only if for every $W \supset Y$ there exists a neighborhood U of f in $C(X, W)$ so that* $\inf\{d_n g(X): g \in U\} > 0$. *($d_n(\cdot)$ is the n-dimensional degree as defined in* [Ku, *p.* 105].)

Proposition 2.3. *Let $f: X \to Y$ be k-unstable. Then for every $g: Y \to Z$ $h = gf: X \to Z$ is k-unstable.*

Proof. We may assume that Y is compact. Let ε be positive. Let Y_1 be a Banach space which contains Y. Let Y_2 denote the closed convex hull of Y in Y_1. Y_2 is compact. Let Z_1 be a Banach space that contains Z and let $\hat{g}: Y_2 \to Z_1$ extend g. Let $\delta > 0$ be so small that $\|y_1 - y_2\| < 3\delta$ in Y_2 implies that $\|\hat{g}(y_1) - \hat{g}(y_2)\| < \varepsilon$ in Z_1. As f is k-unstable there exists some Banach space Y_3 containing Y_1 and a map $f': X \to Y_3$ with $\|f - f'\| < \delta$ and $\dim f'(X) \leq k-1$. By a standard argument there exists a 2δ-translation f'' of $f'(X)$ into some $(k-1)$-dimensional polyhedron H contained in Y_2. (Just pick the vertices of H in Y.) Then $\|f - f''f'\| < 3\delta$ and by the choice of δ $\|gf - \hat{g}f''f'\| = \|\hat{g}f - \hat{g}f''f'\| < \varepsilon$. As $H = f''f'(X)$ is $(k-1)$-dimensional there exists some map $l: H \to Z_1$ with $\dim l(H) \leq k-1$ and $\|l - \hat{g}\| < \varepsilon$ on H. Then $\dim lf''f'(X) \leq k-1$ and $\|h - lf''f'\| \leq \|h - \hat{g}f''f'\| + \|\hat{g}f''f' - lf''f'\| < \varepsilon + \varepsilon = 2\varepsilon$, so h is k-unstable. □

Proposition 2.3 implies in particular that a map which factors through some $(k-1)$-dimensional spaces is k-unstable. It is thus natural to ask whether every k-unstable map must factor through some $(k-1)$-dimensional space. The answer is negative for $k \geq 2$ and affirmative for $k = 1$. (The domain is assumed to be compact.)

Proposition 2.4*. *A* 1*-unstable map factors through some* 0*-dimensional space. For $k \geq 2$ there exist k-unstable maps which do not factor through any $(k-1)$-dimensional space.*

Proposition 2.5*. *A light map on a k-dimensional space is k-stable.*

Actually much stronger results hold:

Theorem 2.1*. *Let $f: X \to Y$ be a map and let $Y_k = \{y \in Y: \dim f^{-1}(y) \geq k\}$. If $\dim X > \max\{k + \min\{n-1, \dim Y_k\}, k = 0, 1, \ldots, \dim f\}$ then f is n-stable. (This is a "stable" version of Vainstein's theorem* [E_2, *p.* 283].) *Theorem* 2.1 *implies in particular*

Proposition 2.6*. *Let $f: X \to Y$ be a map. If*

$$\dim X > \dim f + \dim\{y \in Y: \dim f^{-1}(y) > \dim X - n\}$$

then f is n-stable.

This can be applied to prove the following Chogoshvili-type theorem.

Theorem 2.2*. *Let X be an n-dimensional compact subset of E_m. Then one of the $(m-1)$-dimensional coordinate projections is n-stable on X.*

3. The lattice of upper semicontinuous decompositions

In this section we assume that X is a compact space, and that the range of the maps we consider on X is metrizable. We let $f: X \to Y$ be a map. Then f induces an upper semicontinuous (u.s.c.) decomposition $\{f^{-1}(y): y \in f(X)\}$ on X (see [Ku, p. 65]) and conversely, the quotient map of each u.s.c. decomposition of X has a compact metrizable range.

Let f and g be two maps on X. We let $f \sim g$ if f and g induce the same u.s.c. decomposition on X. Evidently this occurs if and only if there exists a homeomorphism $h: f(X) \to g(X)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & f(X) \\ & \searrow^{g} & \downarrow{h} \\ & & g(X) \end{array}$$

commutes.

Let $\mathrm{DEC}(X)$ denote the set of all maps on X modulo the equivalence relation $\sim$. Clearly the elements of $\mathrm{DEC}(X)$ can be regarded as u.s.c. decompositions of X, but we shall still use functional notation. For f, g in $\mathrm{DEC}(X)$ set $f \leq g$ if the decomposition of X induced by f refines that of g. This is equivalent to the existence of an h which is not necessarily a homeomorphism in the above diagram. In that case we say that f refines g or that g is coarser that f.

It follows from Proposition 2.3 that if $f \leq g$ and if f is k-unstable then so is g. In particular, $f: X \to Y$ is k-unstable if and only if all the maps in the equivalence class mod $\sim$ of f are k-unstable. For f, g in $\mathrm{DEC}(X)$ let $f \wedge g = h$ be the decomposition of X which, at $x \in X$ is the intersection of the fibers of f and g at x. Thus $h^{-1}h(x) = f^{-1}f(x) \cap g^{-1}g(x)$. As a map h may be regarded as $h: X \to f(X) \times g(X)$ with $h(x) = (f(x), g(x))$. Similarly we define the inf of any set $\{f_a\}_{a \in A} \subset \mathrm{DEC}(X)$ by $f = \bigwedge_a f_a$, $f^{-1}f(x) = \bigcap_a f_a^{-1} f_a(x)$, and f can be regarded as a map from X into $\pi_a f_a(X)$ whose a coordinate is f_a. Clearly $\bigwedge_a f_a \leq f_a$ for all a and if $g \in \mathrm{DEC}(X)$

satisfies $g \leq f_a$ for all a then $g \leq \bigwedge_a f_a$. So $\bigwedge$ is actually an inf. We may define $\bigvee F$ for $F \subset \mathrm{DEC}(X)$ as $\bigvee F = \bigwedge\{g: f \leq g \text{ for all } f \text{ in } F\}$. $\mathrm{DEC}(X)$ with these operations is a complete lattice. Note however that unlike the $\bigwedge$ operation which has a simple explicit representation in terms of intersections, the structure of $f \vee g$ is not clear at all. In the next section we shall see that under certain restrictions $f \vee g$ also has a simple structure.

Fix a positive integer k. For $f \in \mathrm{DEC}(X)$ define

Definition 3.1. $f^* = \{g: g \in \mathrm{DEC}(X),\ g \leq f,\ g \text{ is } k\text{-unstable}\}$. By Proposition 2.3 f^* is empty if and only if f is k-stable.

Proposition 3.1. *Let f, $g \in \mathrm{DEC}(X)$. $f \wedge g$ is k-stable if and only if $f^* \cap g^* = \varnothing$.*

Proof. If $f \wedge g$ is k-stable then $f^* \cap g^* = \varnothing$ since if $h \in f^* \cap g^*$ then $h \leq f \wedge g$ is k-unstable and $f \wedge g$ must be k-unstable too. If $f^* \cap g^* = \varnothing$ then $f \wedge g$ must be k-stable since otherwise $f \wedge g$ would be in $f^* \cap g^*$. □

The results in the rest of this section are not applied in the construction.

Proposition 3.2*. *If f is k-unstable then f^* contains minimal elements, i.e. elements g so that $h \leq g$, $h \neq g$ implies that h is k-stable.*

Let U be a (finite open) cover of X. For $f \in \mathrm{DEC}(X)$ set $f \leq U$ if f refines U.

Theorem 3.1*. *f is k-unstable if and only if every cover U such that $f \leq U$ has a refinement V of order $\leq k$.*

Remark. Note that it is not required that $f \leq V$. (This would imply that $\dim f(X) < k$.)

Proposition 3.3*. *For $k > 1$ there exist minimal k-unstable elements of f of $\mathrm{DEC}(X)$ with $\dim f(X) = k$. (Compare with Proposition 2.4*.)*

The families $U = \{f \in \mathrm{DEC}(X): f \leq U\}$, U a cover of X, form a basis for a topology on $\mathrm{DEC}(X)$. This topology is rather trivial. (Note that the class of the embeddings is an element of very open set.) However

Theorem 3.2*. *On the set of minimal k-unstable elements of $\mathrm{DEC}(X)$ this topology is a Hausdorff topology. (Note that if $\dim X < k$ then there is only one minimal k-unstable element namely the class of the embeddings.)*

4. The lattice of monotone u.s.c. decompositions of a hereditarily indecomposable continuum

In this section X is assumed to be an H.I. continuum.

Thus if F and H are subcontinua of X so that $H \cap F \neq \varnothing$ then either $H \subset F$ or $F \subset H$. (Since otherwise $H \cup F$ would be a decomposable continuum.) It follows that

Proposition 4.1. *Every family of subcontinua of X with a nonempty intersection is totally ordered by inclusion.*

An element f of $\mathrm{DEC}(X)$ is monotone if $f^{-1}(y)$ is a continuum for all y in $f(X)$. Let $M(X)$ denote the set of all monotone elements of $\mathrm{DEC}(X)$. In general $M(X)$ is not a sublattice of $\mathrm{DEC}(X)$ but when X is H.I. then it is.

Proposition 4.2. *Let* $\{f_a\}_{a\in A} \subset M(X)$. *Then* $f = \bigwedge_a f_a \in M(X)$.

Proof. $f^{-1}f(x) = \bigcap_a f_a^{-1}f_a(x)$. All the continua $f_a^{-1}f_a(x)$ contain x. Thus by Proposition 4.1 they are totally ordered by inclusion and hence their interaction is a continuum. □

Definition 4.1. For f, g in $M(X)$ let $h = f \vee g$ be defined by $h^{-1}h(x) = f^{-1}f(x) \cup g^{-1}g(x)$.

Proposition 4.3. $h = f \vee g$ *is a well-defined element of* $M(X)$ *and it agrees with the earlier definition of* $f \vee g$ *in* DEC(X) *namely* $\bigwedge\{l \colon f \le l,\ g \le l\}$.

Proof. By Proposition 4.1 $h^{-1}h(x)$ is the larger among $f^{-1}f(x)$ and $g^{-1}g(x)$. It follows that 4.1 actually defines a closed decomposition of X. We check that it is u.s.c. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of fibers of h which converges to some elements u in 2^X. We must show that u is contained in some fiber of h.

Each u_n is either a fiber of f or a fiber of g and as f and g are u.s.c. u must be contained in a fiber of f or in a fiber of g. So, if $x \in u$ then $u \subset f^{-1}f(x)$ or $u \subset f^{-1}g(x)$ and thus $u \subset f^{-1}f(x) \cup g^{-1}g(x) = h^{-1}h(x)$, and it follows that h is u.s.c. Thus $h \in M(X)$. If $f \le l$ and $g \le l$ then $f \vee g \le l$, also $f \le f \vee g$ and $g \le f \vee g$. It follows that $h = \bigwedge\{l \colon l \in \text{DEC}(X),\ f \le l,\ g \le l\}$. □

Definition 4.2. For $f \in \text{DEC}(X)$ let the singular set S_f of f be defined by $S_f = \{x \in X \colon f^{-1}f(x) = \{x\}\}$.

Proposition 4.4. *Let* f, $g \in M(X)$. *Then* $S_{f\wedge g} = S_f \cup S_g$ *and* $S_{f\vee g} = S_f \cap S_g$.

Proof. Let $h = f \wedge g$. $h^{-1}h(x) = f^{-1}f(x) \cap g^{-1}g(x)$ so clearly $h^{-1}h(x) = \{x\}$ if and only if either $f^{-1}f(x) = \{x\}$ or $g^{-1}g(x) = \{x\}$ (recall that one of $f^{-1}f(x)$ and $g^{-1}g(x)$ contains the other) and it follows that $S_h = S_f \cup S_g$.

Let $l = f \vee g$. $l^{-1}l(x) = f^{-1}f(x) \cup g^{-1}g(x)$ and thus $l^{-1}l(x) = \{x\}$ if and only if both $f^{-1}f(x) = \{x\}$ and $g^{-1}g(x) = \{x\}$ so $S_l = S_f \cap S_g$. □

Remark. The fact that $M(X)$ is a sublattice of DEC(X) and in particular the simple structure of $f \vee g$ as reflected in Definition 4.1 and Propositions 4.3 and 4.4 are the main advantages of H.I. spaces that are applied in the construction.

5. A counterexample to the Chogoshvili conjecture

Let $n \ge 2$ and $s \ge 2n - 1$ be integers and let k be the greatest integer $\le (s-1)/(n-1)$ (so $k \ge 2$). Let $m = \binom{s}{s-n+1}(4(s-n)+3)$. Let X be an n-dimensional H.I. space (which exists by [B]). We prove the following:

Theorem 5.1. *There exists an embedding* $g \colon X \to E_m$ *and a decomposition* $E_m = \sum_a \bigoplus E_a$, *of* E_m *into an orthogonal direct sum of* $\binom{s}{s-n+1}$ *linear subspaces* E_a *of dimension* $l = 4(s-n)+3$ *each, such that for every choice of* k *of the subspaces* E_a, $\{E_{a_i}\}_{i=1}^{k}$ *the composition* Pg *of* g *with the canonical projection* $P \colon E_m \to \sum_{i=1}^{k} \bigoplus E_{a_i}$ *is 2-unstable.*

Theorem 1.1 follows directly from Theorem 5.1 and the results of §2. For $k > 2$ we obtain more information that in Theorem 1.1, namely large subspaces (sums of kE_a's) with large overlaps such that the projection of $g(X)$ on them is 2-unstable. Note however that by Theorem 2.2* the projection of $g(X)$ onto one of the $(m-1)$-dimensional coordinate subspaces of E_m must be n-stable.

Proof of Theorem 5.1. As by a theorem of Hurewicz the light mappings form a dense G_δ set in $C(X, R^n)$ [Ku, p. 125] we can find in $C(X, R^s)$ an element $f' = (f_1', f_2', \dots, f_s')$ so that for every $1 \le i_1 < i_2 < \cdots < i_n \le s$ the element $f' = (f_{i_1}', f_{i_2}', \dots, f_{i_n}')$ of $C(X, R^n)$ is light. For each $1 \le i \le s$ let

$$\begin{array}{ccc} X & \xrightarrow{f_i} & Z_i \\ & \searrow^{f_i'} & \downarrow l_i \\ & & R \end{array}$$

denote the monotone-light decomposition of f_i' (see [Ku, p. 184]). As l_i is light it follows from the Hurewicz Theorem [H.W, p. 91] that $\dim Z_i \le \dim R = 1$; and as f_i is monotone and has a one-dimensional range f_i is a 2-unstable element of $M(X)$. Let $1 \le i_1 < i_2 < \cdots < i_n \le s$. As $f_i \le f_i'$, $\bigwedge_{1\le j\le n} f_{i_j} \le \bigwedge_{1\le j\le n} f_{i_j}'$ and since the latter map is light so is the first. Hence $\bigwedge_{1\le j\le n} f_{i_j}$ is both monotone and light and thus an embedding. Let $S_i = S_{f_i}$ (see Definition 4.2). From Proposition 4.4 it follows that

5.1.

$$X = S_{\bigwedge_{1\le j\le n} f_{i_j}} = \bigcup_{j=1}^{n} S_{i_j}.$$

So we proved

5.2. The union of every n of the S_i's is X. It follows that

5.3. Each point x of X belongs to S_i for at least $s - n + 1$ values of i. Indeed, let $x \in X$. If x belongs only to $s - n$ S_i's then the remaining n S_i's would not cover X violating 5.2.

Let $A = \{a \subset \{1, 2, \dots, s\}$, cardinality of $a = |a| = s - n + 1\}$. 5.3 is equivalent to

5.4. $X = \bigcup_{a\in A} \bigcap_{i\in a} S_i$.

We claim that

5.5. Every k elements of A have a nonempty intersection.

Proof.

$$\left|\{1, 2, \dots, s\} \setminus \bigcap_{i=1}^{k} a_i\right| = \left|\bigcup_{i=1}^{k}(\{1, 2, \dots, s\} \setminus a_i)\right| \le \sum_{i=1}^{k} |\{1, 2, \dots, s\} \setminus a_i|$$

$$= k(n-1) \le \frac{s-1}{n-1}(n-1) = s - 1.$$

Hence $\bigcap_{i=1}^{k} a_i$ must contain at least one element. □

For $a \in A$ let $\psi_a \in M(X)$ be defined by $\psi_a = \bigvee_{i\in a} f_i$ (see Definition 4.1). Let also $Y_a = \psi_a(X)$ and $\psi = \bigwedge_{a\in A} \psi_a : X \to \prod_{a\in A} Y_a = Y$.

5.6. *Claim* ψ is an embedding.

Proof. By Proposition 4.4 and by 5.4, $S_\psi = \bigcup_{a\in A} S_{\psi_a} = \bigcup_{a\in A} \bigcap_{i\in a} S_i = X$. □

5.7. *Claim.* Let $a_1, a_2, \dots, a_k$ be k elements of A. Then $\bigwedge_{1\le j\le k} \psi_{a_j}$ is 2-unstable. In particular, for a, b in $A \psi_a \wedge \psi_b$ is 2-unstable.

Proof. As $\psi_a = \bigvee_{i\in a} f_i$ and since the f_i's are 2-unstable $\{f_i\}_{i\in a} \subset \psi_a^*$ (see Definition 3.1).

By 5.5 there exists some $1 \le i_0 \le s$ so that $i_0 \in a_j$ for all $1 \le j \le k$. Then $f_{i_0} \in \psi_{a_j}^*$ for $1 \le j \le k$ and thus $f_{i_0} \in \bigcap_{1\le j\le k} \psi_{a_j}^*$. By Proposition 3.1 $\bigwedge_{1\le j\le k} \psi_{a_j}$ is 2-unstable.

5.8. *Claim*. For each a in A $\dim Y_a \le 2(s-n)+1$.

Proof. Fix a in A. For $i \in a$ let

$$W_i = \{y \in Y_a \colon \psi_a^{-1}(y) \text{ is a fiber of } f_i\}.$$

Recall that $\psi_a = \bigvee_{i\in a} f_i$, so each fiber of ψ_a is a fiber of one of the f_i's, $i \in a$. It follows that $Y_a = \bigcup_{i\in a} W_i$. Consider $\psi_a^{-1}(W_i) \subset X$. The mappings ψ_a and f_i induce the same decomposition on $\psi_a^{-1}(W_i)$ and hence $W_i = \psi_a \psi_a^{-1}(W)$ and $f_i(\psi_a^{-1}(W_i)$ are homeomorphic. But $f_i \psi_a^{-1}(W_i) \subset Z_i$ and dim $Z_i = 1$. Hence dim $W_i \le 1$. Thus Y_a is the union of $s-n+1$ 1-dimensional sets so dim $Y_a \le 2(s-n+1)-1 = 2(s-n)+1$.

Problem. Can one obtain a better estimate of $\dim Y_a$? In particular can one obtain 1-dimensional Y_a's?

Let us summarize what we obtained by now:

5.9. Given an n-dimensional H.I. continuum X and $s \ge 2n-1$, there exist $\binom{s}{s-n+1}$ monotone maps $\psi_a \colon X \to Y_a$, with dim $Y_a \le 2(s-n)+1$ so that $\psi = \bigwedge_a \psi_a \colon X \to \prod_a Y_a$ is an embedding and such that for every choice $a_1, a_2, \dots, a_k$ of k a's (k = greatest integer $\le \frac{s-1}{n-1}$, $k \ge 2$) $\bigwedge_{1\le j\le k} \psi_{a_j}$ is 2-unstable.

Let now $h_a \colon Y_a \to E_a = R^{4(s-n)+3}$ be an embedding. Set $g_a = h_a \psi_a \colon X \to E_a$ and $g = \bigwedge_{a\in A} g_a \colon X \to \sum_{a\in A} \bigoplus E_a = E_m$ where $m = \binom{s}{s-n+1}(4(s-n)+3)$.

Note that since h_a is an embedding ψ_a and g_a determine the same element of $M(X)$. It follows from 5.6 and 5.7 that g is an embedding of X in E_m, and that if P is the canonical projection of E_m onto the direct sum of any k of the E_a's then Pg is 2-unstable. In particular if L is a two-dimensional linear subspace of E_m which is contained in $\sum_{1\le j\le k} \bigoplus E_{a_j} = E$ then the canonical projection Q of E_m onto L has no stable values on $g(X)$ since Q factors through the projection P of E_m onto E and thus $P \le Q$ in DEC$(g(X))$.

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